

A method for calculating the imaginary part of the Hadamard Elementary function $G^{(1)}$ in static, spherically symmetric spacetimes

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Abstract

Whenever real particle production occurs in quantum field theory, the imaginary part of the Hadamard Elementary function $G^{(1)}$ is non-vanishing. A method is presented whereby the imaginary part of $G^{(1)}$ may be calculated for a charged scalar field in a static spherically symmetric spacetime with arbitrary curvature coupling and a classical electromagnetic field A^μ . The calculations are performed in Euclidean space where the Hadamard Elementary function and the Euclidean Green function are related by $\frac{1}{2}G^{(1)} = G_E$. This method uses a 4^{th} order WKB approximation for the Euclideanized mode functions for the quantum field. The mode sums and integrals that appear in the vacuum expectation values may be evaluated analytically by taking the large mass limit of the quantum field. This results in an asymptotic expansion for $G^{(1)}$ in inverse powers of the mass m of the quantum field. Renormalization is achieved by subtracting off the terms in the expansion proportional to nonnegative powers of m , leaving a finite remainder known as the “DeWitt-Schwinger approximation.” The DeWitt-Schwinger approximation for $G^{(1)}$ presented here has terms proportional to both m^{-1} and m^{-2} . The term proportional to m^{-2} will be shown to be identical to the expression obtained from the m^{-2} term in the generalized DeWitt-Schwinger point-splitting ex-

pansion for $G^{(1)}$. The new information obtained with this present method is the DeWitt-Schwinger approximation for the imaginary part of $G^{(1)}$, which is proportional to m^{-1} in the DeWitt-Schwinger approximation for $G^{(1)}$ derived in this paper.

I. INTRODUCTION

Despite the absence of a full quantum theory of gravity, there are well known methods to calculate useful quantities involving quantum fields in curved space. One of the most successful of these methods is the semiclassical approach, wherein the gravitational and electromagnetic fields are treated classically while the various matter fields that act as sources are treated quantum mechanically. In this approach, the evolution of the spacetime from a set of given initial conditions is described by the combined Einstein-Maxwell Field Equations,

$$G_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle, \quad (1)$$

and

$$F^{\mu\nu}{}_{;\nu} = 4\pi\langle j^\mu\rangle, \quad (2)$$

where $\langle T_{\mu\nu}\rangle$ and $\langle j^\mu\rangle$ are the vacuum expectation values (VEVs) of the stress-energy tensor and the current due to a charged quantized scalar field. This paper follows the sign conventions of Misner, Thorne, and Wheeler [1] and uses natural units ($G = c = \hbar = 1$) throughout.

In the semiclassical regime, the calculation of quantities such as $\langle\phi^2\rangle$ and $\langle T_{\mu\nu}\rangle$ may begin with calculation of the Feynman Green function $G_F(x, x')$ defined by [2]

$$iG_F(x, x') = \langle 0|T[\phi(x)\phi(x')]|0\rangle, \quad (3)$$

where $T[\dots]$ is the time ordering operator, $\phi(x)$ is the quantum field evaluated at the stationary spacetime point x , and $\phi(x')$ is the quantum field evaluated at the nearby point x' . One of the major difficulties in using the semiclassical method in quantum field calculations is the fact that infinities appear in the VEVs in Eqs.(1)-(2) in both flat [3–5] and curved [2,6] spacetimes. Various methods have been used to isolate those infinities and remove them from the physical theory [2], yet arguably the most powerful of these methods is based on the classic work of Schwinger [7].

Schwinger calculated the Feynman Green function for a quantized fermion field by introducing a fictitious, non-quantum mechanical Hilbert space. This $(4+1)$ -dimensional Hilbert space was constructed from the 4 spacetime dimensions in addition to a fifth dimension which was identified as the proper time parameter s within this fictitious space. Working within this proper time space, Schwinger was able to both isolate the divergences in the quantum field integrals for the fermion current $\langle j^\mu(x) \rangle = \lim_{x' \rightarrow x} i \text{etr}[\gamma^\mu G_F(x, x')]$ produced by an external electromagnetic field in flat spacetime, and to use those divergences to renormalize the charge e of the quantum field.

Schwinger's calculations of the quantum action functional W were performed by transforming the integrals involving the 4 spacetime dimensions into the momentum representation. The integrals were evaluated using perturbation expansions in powers of eA_μ and $eF_{\mu\nu}$, where A_μ and $F_{\mu\nu}$ are the gauge field vector and the electromagnetic field tensor, respectively. Were these integrals to be evaluated without further modification, Schwinger noted [8] that conservation of energy and momentum would dictate that no pair creation would occur, or $\langle j^\mu \rangle = 0$. Schwinger's solution to this situation was to add an infinitesimal imaginary part to the action integral, resulting in W acquiring a positive imaginary part. Schwinger associated this imaginary part of W with pair production by stating that

$$\left| e^{iW} \right|^2 = e^{-2\text{Im}[W]} \quad (4)$$

represented the probability that no pair creation would occur. After this identification was made, Schwinger's calculation of the pair creation rate yielded a series expansion for the probability of pair creation per unit four volume Γ by a constant external electric field,

$$\Gamma = \frac{e^2}{4\pi^3} E^2 \sum_{n=1}^{\infty} n^{-2} e^{\left(\frac{-n\pi m^2}{eE} \right)}, \quad (5)$$

where m and e are the mass and charge of the fermion field, respectively, and E is the magnitude of the electric field. The expansion of Eq.(5) yields a series involving inverse powers of m^2 .

Schwinger's calculation for the pair creation rate assumed that spacetime was flat. Building on the work of Schwinger, DeWitt [9,6] extended the proper time method to include

curved spacetime results. Using what is now called the DeWitt-Schwinger proper time method, DeWitt calculated an asymptotic expansion for the Feynman Green function for a real scalar field in an arbitrary curved spacetime. This asymptotic expansion was an expansion in inverse powers of m^2 , where m is the mass of the quantum field. This expansion of $G_F(x, x')$ yielded an expression with both real and imaginary parts. The real and imaginary parts of $G_F(x, x')$ could then be identified with other Green functions by using the identity

$$G_F(x, x') = \overline{G}(x, x') - \frac{1}{2}iG^{(1)}(x, x'), \quad (6)$$

where $\overline{G}(x, x')$ is one-half the sum of the advanced and retarded Green functions, and the Hadamard Elementary function $G^{(1)}(x, x')$ [2] is defined by

$$G^{(1)}(x, x') \equiv \langle 0 | \{ \phi(x), \phi(x') \} | 0 \rangle. \quad (7)$$

Thus, the DeWitt-Schwinger asymptotic series expansion for $G^{(1)}(x, x')$ has the general form,

$$G^{(1)}(x, x') \sim B_{+2}(x, x')m^2 + B_0(x, x')m^0 + B_{-2}(x, x')m^{-2} + \dots, \quad (8)$$

where the B_n are coefficients constructed from curvature tensors.

In the semiclassical method, quantities such as $\langle \phi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ may be calculated from $G^{(1)}(x, x')$ by taking the appropriate derivatives with respect to either the stationary space-time point x or the nearby point x' [6,9–11,2]; e.g.,

$$\langle \phi^2 \rangle = \lim_{x' \rightarrow x} \frac{1}{2} G^{(1)}(x, x'), \quad (9)$$

and

$$\langle T^{\mu\nu}(x) \rangle = \lim_{x' \rightarrow x} \text{Re} \left[\frac{1}{2} \left(\frac{1}{2} - \xi \right) (g^\mu{}_{\tau'} G^{(1)|\tau'\nu} + g^\nu{}_{\rho'} G^{(1)|\mu\rho'}) + \left(\xi - \frac{1}{4} \right) g^{\mu\nu} g^{\alpha\rho'} G^{(1)}_{|\alpha\rho'} \dots \right]. \quad (10)$$

In Eq.(10), the vertical bar “|” indicates gauge covariant differentiation, and $g^\mu{}_{\tau'}$ is the bivector of parallel transport [9], which is used to transport the information from the nearby point x' to the stationary point x . The infinities appearing in the VEV in Eq.(9) lead to the well known divergences in the unrenormalized expression for $\langle T^{\mu\nu}(x) \rangle$ in Eq.(10)

[2]. DeWitt’s asymptotic expansion of $G^{(1)}(x, x')$ in powers of m^2 isolated the infinities, or the “counterterms,” of both $G^{(1)}(x, x')$ and $\langle T_{\mu\nu} \rangle$, with the infinities being found as coefficients of the m^2 and m^0 terms in the series of Eq.(8) [10]. Once these infinities have been subtracted from the unrenormalized VEVs in Eqs.(9)-(10), then the finite remainder carries the information about the physics of the spacetime. While these subtractions may be performed in principle, the remaining expressions are often so complicated that they can not be evaluated analytically [2].

Investigators have used the semiclassical method to calculate renormalized expressions for both the vacuum polarization $\langle \phi^2 \rangle_{ren}$ [12–17], and the stress energy tensor $\langle T_{\mu\nu} \rangle_{ren}$ [18–21] in specific spacetimes. In each of these cases, the point-splitting counterterms were used to renormalize the divergent VEVs. These calculations were performed in spacetimes whose high degrees of symmetry allowed the subtractions of the counterterms to be evaluated analytically. See Ref. [21] for a more thorough discussion of the difficulties encountered in these subtractions.

The DeWitt-Schwinger point-splitting expansion for $G^{(1)}$ may also do more than simply isolating the divergent counterterms necessary to renormalize the VEVs encountered in semiclassical quantum field theory. If the terms proportional to nonnegative powers of m are discarded, the remaining terms in Eq.(8) constitute the “DeWitt-Schwinger approximation” for $G^{(1)}$ [21–23] for fields whose masses are large when compared to the magnitude of the coefficients B_n . Retaining the first one or two of these terms has been shown to provide a close approximation to the exact results in several cases [14,19,17,21].

In the case of charged scalar fields, $G^{(1)}$ involves the complex quantity

$$G^{(1)}(x, x') \equiv \langle 0 | \{ \phi(x), \phi^*(x') \} | 0 \rangle, \quad (11)$$

where $\phi(x)$ is the charged scalar field. Even though $G^{(1)}$ is generally complex, since all of the coefficients B_n are real, Eq.(8) only contains part of the information about $G^{(1)}$. DeWitt has pointed out [6] that the point-splitting asymptotic expansion of $G^{(1)}$ in inverse powers of m^2 is incapable of yielding the imaginary part of $G^{(1)}$. Yet whenever real particle production

occurs $G^{(1)}$ will have a non-zero imaginary part. This is readily seen by examining the expression for the current due to a charged scalar field,

$$j^\mu = \frac{1}{4\pi} F^{\mu\nu}{}_\nu = \frac{ie}{2} [\{D^\mu \phi, \phi^*\} - \{D^\mu \phi, \phi^*\}^*], \quad (12)$$

where e is the charge of the field, $D^\mu \equiv (\nabla^\mu - ieA^\mu)$ is the gauge covariant derivative, A^μ is the background gauge field, and the asterisk $*$ denotes the complex conjugate. Using Eq.(11), and making the transition from classical to quantum fields [24], gives the vacuum expectation value of the current due to the coupling between the charged scalar field and the background gauge field,

$$\langle j^\mu(x) \rangle = \lim_{x' \rightarrow x} \frac{ie}{4} \left[\left(G^{(1)|\mu} + g^\mu{}_{\tau'} G^{(1)|\tau'} \right) - \left(G^{(1)|\mu} + g^\mu{}_{\tau'} G^{(1)|\tau'} \right)^* \right]. \quad (13)$$

Since the coefficients B_n in Eq.(8) are real, using the DeWitt-Schwinger point-splitting expansion involving only inverse powers of m^2 to construct the current $\langle j^\mu \rangle$ from Eq.(13) can only yield a value of zero. Yet the current is not identically zero, as evidenced by Schwinger's calculation in flat space, thus illustrating the limitation the DeWitt-Schwinger point-splitting expansion suffers by yielding only the imaginary part of $G^{(1)}$.

In this paper, a method is presented to obtain an analytic expression for the imaginary part of $G^{(1)}$ in a static spherically symmetric spacetime with a gauge field A^μ . The imaginary part arises in a straightforward manner due to the presence of the gauge field A^μ and does not require the addition of any extra terms during the course of the calculation. The renormalized expression $G_{ren}^{(1)}$ would ordinarily be constructed by subtracting the DeWitt-Schwinger point-splitting counterterms from the unrenormalized expression for $G^{(1)}$;

$$G_{ren}^{(1)} = G_{unren}^{(1)} - G_{DS}^{(1)}. \quad (14)$$

The mode sums and integrals in Eq.(14) can not be evaluated analytically at present due to the complexity of the terms in the infinite sums and integrals. A method recently described by Anderson, Hiscock, and Samuel (AHS) [21] yields a way to construct an approximate expression for $G_{ren}^{(1)}$. All subtractions are performed in Euclidean space where the relationships between various Green functions are given by [17]

$$\langle \phi^2 \rangle = \frac{1}{2} G^{(1)}(x, x') = iG_F(x, x') = G_E(x, x'). \quad (15)$$

Using the Euclideanized metric, the mode functions in the VEV of Eq.(11) may be rewritten in a WKB form [17,21]. The renormalized expression $G_{E,ren}$ is constructed by moving into Euclidean space and performing the subtraction

$$G_{E,ren} = G_{E,WKB} - G_{E,WKBdiv}, \quad (16)$$

where $G_{E,WKB}$ is the expression obtained in Euclidean space upon substituting the Euclidean WKB mode functions into Eq.(11), and $G_{E,WKBdiv}$ contains the ultraviolet divergences found in $G_{E,WKB}$. The mode sums and integrals in Eq.(16) still can not be evaluated analytically in their present form. In order to construct an expression for $G_{E,ren}$, the fourth order WKB approximations for the exact mode functions are substituted into the quantum field expressions. Then, the mode sums and integrals in the subtraction of Eq.(16) may be evaluated analytically by expanding the integrands and summands in powers of the mass m of the quantum field. This mass is assumed to be large when compared to the inverse of the radius of curvature of the spacetime. By working within this large mass limit, a DeWitt-Schwinger asymptotic expansion results for $G_{E,ren}$ similar to Eq.(8);

$$G_{E,ren} = (G_{E,WKB} - G_{E,WKBdiv})_{large\ m} \quad (17)$$

$$\sim B_{+2}(x, x')m^2 + B_{+1}(x, x')m^1 + B_0(x, x')m^0 \quad (18)$$

$$+ B_{-1}(x, x')m^{-1} + B_{-2}(x, x')m^{-2} + \dots \quad (19)$$

Eq.(19) differs from Eq.(8) in that the presence of the gauge field in this method is directly responsible for the presence of the new terms involving odd powers of m . Renormalization is achieved by discarding terms proportional to nonnegative powers of m , leaving a finite DeWitt-Schwinger approximation for $G_{E,ren}$;

$$G_{E,ren} \approx B_{-1}(x, x')m^{-1} + B_{-2}(x, x')m^{-2}. \quad (20)$$

In Euclidean space, all of these terms are real. When the final expression is rotated back to the Lorentzian sector, the real and imaginary parts of $G^{(1)}$ are obtained. The terms

proportional to m^{-1} are all imaginary and are proportional to both odd powers of the gauge field A^μ and its derivatives along with odd powers of the charge e of the quantum field. The terms proportional to m^{-2} are all real and are exactly those obtained from the m^{-2} term of the DeWitt-Schwinger point-splitting expansion [10,17,22].

In Sec. II, an expression is derived for the unrenormalized Euclidean Green function G_E in a static spherically symmetric spacetime. In this paper, the field is assumed to be at zero temperature, although this method would extend to the analysis of fields at nonzero temperature [17]. Sec. III renormalizes the expression for G_E derived in Sec. II. The renormalization subtractions are evaluated analytically by evaluating the mode sums and integrals in the unrenormalized expression in the limit that the mass of the quantum field is large when compared to the inverse of the radius of curvature of the spacetime. The resulting DeWitt-Schwinger approximation will be shown to contain both the real and imaginary parts of $G^{(1)}$, with the main goal of this paper being obtaining an expression for the imaginary part.

Sec. IV contains the results of the calculation of the asymptotic expansion of $G^{(1)}$. The terms proportional to m^{-2} will be shown to give the same results as the m^{-2} term in the generalized DeWitt-Schwinger point-splitting expansion when the general expression is evaluated in a static spherically symmetric spacetime [10,17]. The terms proportional to m^{-1} are all real in Euclidean space but are all imaginary when rotated back to Lorentzian space. All of the m^{-1} terms are proportional to odd powers of the charge e of the complex scalar field, as well as odd powers of the gauge field A^μ and derivatives, and thus these terms all vanish when either the quantum field is uncharged or the spacetime is uncharged. This connection to previous work will serve as a check on the validity of the method presented here.

II. UNRENORMALIZED GREEN FUNCTIONS FOR THE CHARGED SCALAR FIELD IN EUCLIDEAN SPACE

The goal of this paper is to derive an expression for the imaginary part of $G^{(1)}$ so that curved space phenomena involving real particle production due to the presence of an electromagnetic field may be investigated. Thus it is useful to begin with the electromagnetic vector potential for a static spherically symmetric electric field like that of the Reissner-Nordström spacetime;

$$A^\mu = (A^t(r), 0, 0, 0). \quad (21)$$

The calculations will be simplified by moving to Euclidean space [17]. The Euclidean metric for a static spherically symmetric spacetime is

$$ds^2 = f(r)d\tau^2 + h(r)dr^2 + r^2d\Omega^2, \quad (22)$$

where $\tau \equiv -it$, $f(r)$ and $h(r)$ are the same functions as in the Lorentzian metric for a static spherically symmetric spacetime

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2d\Omega^2, \quad (23)$$

and t is the coordinate whose derivative $(\partial/\partial t)$ is the timelike Killing vector in the Lorentzian sector [17]. The Lorentzian and Euclidean expressions for the invariant $A_\mu A^\mu$ are given by

$$\begin{aligned} A_\mu A^\mu &= g_{tt}A_L^t A_L^t = -f(r)A_L^t A_L^t = f(r)(iA_L^t)(iA_L^t) \\ &= g_{\tau\tau}A_E^\tau A_E^\tau = f(r)A_E^\tau A_E^\tau, \end{aligned} \quad (24)$$

where the subscripts “L” and “E” refer to the Lorentzian and Euclidean sectors, respectively. Eq.(24) indicates that

$$A_E^\tau = iA_L^t. \quad (25)$$

This shows the relation between the gauge invariant derivative operators in both sectors is

$$\begin{aligned}
g_{\mu\nu}D_L^\mu D_L^\nu &= g_{tt}(\nabla^t - ieA^t)(\nabla^t - ieA^t) + \dots \\
&= -f(r)(\nabla^t - ieA^t)(\nabla^t - ieA^t) + \dots \\
&= f(r)(i\nabla^t - ie(iA^t))(i\nabla^t - ie(iA^t)) + \dots \\
&= f(r)(\nabla^\tau - ieA^\tau)(\nabla^\tau - ieA^\tau) + \dots \\
&= g_{\tau\tau}D^\tau D^\tau + \dots \\
&= g_{\mu\nu}D_E^\mu D_E^\nu
\end{aligned} \tag{26}$$

indicating

$$D^\tau \equiv \nabla^\tau - ieA^\tau \tag{27}$$

for the Euclidean sector.

An explicit form for the Euclidean Green function for fields at zero temperature is needed. The case of zero temperature is chosen here for simplicity, and the derivation here follows those of Anderson [17] and AHS [21]. The action of the wave operator on the Euclidean Green function is given by

$$\begin{aligned}
[g_{\mu\nu}D_E^\mu D_E^\nu - (m^2 + \xi R)]G_E(x, x') &= -g^{-1/2}\delta^4(x, x') \\
&= -\frac{\delta(\tau, \tau')\delta(r, r')\delta(\Omega, \Omega')}{r^2(fh)^{1/2}}.
\end{aligned} \tag{28}$$

The delta function $\delta(\Omega, \Omega')$ may be expanded using the Legendre polynomials with the result

$$\delta(\Omega, \Omega') = \frac{1}{4\pi} \sum_{l=0}^{\infty} (2l+1)P_l[\cos(\gamma)], \tag{29}$$

where $\cos(\gamma) = \cos(\theta)\cos(\theta') + \sin(\theta)\sin(\theta')\cos(\phi - \phi')$. With the scalar field chosen to be at zero temperature, the delta function for the split in the $\tau - \tau'$ direction is given by

$$\delta(\tau, \tau') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega(\tau - \tau')}. \tag{30}$$

Thus, the Euclidean Green function for a field at zero temperature is

$$G_E(x, x') = \frac{1}{4\pi^2} \int_0^\infty d\omega \cos[\omega(\tau - \tau')] \sum_{l=0}^{\infty} (2l+1)P_l[\cos(\gamma)]S_{\omega l}(r, r'), \tag{31}$$

where $S_{\omega l}(r, r')$ is an unknown function of the coordinate r .

Applying the wave operator of Eq.(26) to G_E gives a differential equation for $S_{\omega l}(r, r')$,

$$\begin{aligned} \frac{1}{h} \frac{d^2 S}{dr^2} + \left[\frac{2}{rh} + \frac{1}{2fh} \frac{df}{dr} - \frac{1}{2h^2} \frac{dh}{dr} \right] \frac{dS}{dr} \\ - \left[\frac{(\omega - eA_r)^2}{f} + \frac{l(l+1)}{r^2} + m^2 + \xi R \right] S = -\frac{\delta(r, r')}{r^2(fh)^{1/2}}, \end{aligned} \quad (32)$$

where m is the mass of the field, and A_r is the only non-zero component of the Euclideanized vector potential. $S_{\omega l}(r, r')$ is given by

$$S_{\omega l}(r, r') \equiv C_{\omega l} p_{\omega l}(r) q_{\omega l}(r'), \quad (33)$$

where $C_{\omega l}$ is a normalization factor, and $p_{\omega l}(r)$ and $q_{\omega l}(r')$ are the solutions of the homogeneous form of Eq.(32). In Refs. [17] and [21], $S_{\omega l}(r, r')$ is written in the form

$$S_{\omega l}(r, r') \equiv C_{\omega l} p_{\omega l}(r_{<}) q_{\omega l}(r_{>}). \quad (34)$$

$S_{\omega l}(r, r')$ satisfies the Wronskian condition

$$C_{\omega l} \left[p_{\omega l} \frac{dq_{\omega l}}{dr} - q_{\omega l} \frac{dp_{\omega l}}{dr} \right] = -\frac{1}{r^2} \left(\frac{h}{f} \right)^{1/2}, \quad (35)$$

obtained by integrating Eq.(32) once with respect to r from $r - \epsilon$ to $r + \epsilon$, with $\epsilon \rightarrow 0$ in the end.

The functions $p_{\omega l}$ and $q_{\omega l}$ may be put into a WKB form using [17]

$$p_{\omega l}(r) = \frac{1}{\sqrt{2r^2 W(r)}} e^{\int^r W(r') \left(\frac{h}{f} \right)^{1/2} dr'} \quad (36)$$

$$q_{\omega l}(r) = \frac{1}{\sqrt{2r^2 W(r)}} e^{-\int^r W(r') \left(\frac{h}{f} \right)^{1/2} dr'}. \quad (37)$$

In Eq.(36), $p_{\omega l}(r)$ is well behaved at $r = 0$ and the event horizons, but is divergent at $r = \infty$. The function $q_{\omega l}(r)$ is divergent at $r = 0$ and the event horizons, but is well behaved at $r = \infty$. See Anderson [17] for a more complete discussion of the properties of these functions. Eqs.(35)–(37) show that $C_{\omega l} = 1$. Using Eqs.(36) and (37), Eq.(32) assumes the familiar WKB form

$$W^2 = \Omega^2(r) + V_1(r) + V_2(r) + \frac{1}{2} \left[\frac{f}{hW} \frac{d^2 W}{dr^2} + \left(\frac{1}{h} \frac{df}{dr} - \frac{1}{fh^2} \frac{dh}{dr} \right) \frac{1}{2W} \frac{dW}{dr} - \frac{3}{2} \frac{f}{h} \left(\frac{1}{W} \frac{dW}{dr} \right)^2 \right], \quad (38)$$

where

$$\Omega^2(r) \equiv (\omega - eA_\tau)^2 + m^2 f + (l + \frac{1}{2})^2 \frac{f}{r^2}, \quad (39)$$

$$V_1(r) \equiv \frac{1}{2rh} \frac{df}{dr} - \frac{f}{2rh^2} \frac{dh}{dr} - \frac{f}{4r^2}, \quad (40)$$

$$V_2(r) \equiv \xi R f = -\xi f \times \left[\frac{1}{fh} \frac{d^2 f}{dr^2} - \frac{1}{2f^2 h} \left(\frac{df}{dr} \right)^2 - \frac{1}{2fh^2} \frac{df}{dr} \frac{dh}{dr} + \frac{2}{rfh} \frac{df}{dr} - \frac{2}{rh^2} \frac{dh}{dr} + \frac{2}{r^2 h} - \frac{2}{r^2} \right], \quad (41)$$

and where the product $l(l+1)$ has been factored into two pieces according to

$$l(l+1) = (l + \frac{1}{2})^2 - \frac{1}{4}. \quad (42)$$

The simple form of the vector potential in Eq.(21) causes the WKB equation to assume a new but straightforward form. Previous work had WKB equations just like Eqs.(38)-(41) with $A_\tau = 0$ (see the references in Ref. [21] for a more detailed list of work by others). The new feature in the WKB differential equations is the presence of the electromagnetic vector potential. Eq.(38) indicates how the complex quantities necessary for determining the charged scalar field current $\langle j^\mu \rangle$ arise in the present method. The WKB function $W(r)$, while real in the Euclidean sector, is now complex in Lorentzian space due to the electromagnetic field of Eq.(25).

Eq.(38) is not in general exactly solvable. Fortunately, it may be solved approximately by iteration. For example, the zeroth order solution for W is $W = \Omega$. Substituting $W \rightarrow \Omega$ into the right hand side of Eq.(38) and solving for W yields the second order WKB solution $W^{(2)}$,

$$W^{(2)} = \Omega + \frac{1}{2\Omega}(V_1 + V_2) + \frac{1}{4\Omega^2}V_1^2 + \frac{1}{4} \left[\frac{f}{h\Omega^2} \frac{d^2 \Omega}{dr^2} + \left(\frac{1}{h} \frac{df}{dr} - \frac{f}{h^2} \frac{dh}{dr} \right) \frac{1}{2\Omega^2} \frac{d\Omega}{dr} - \frac{3}{2} \frac{f}{h} \frac{1}{\Omega^3} \left(\frac{d\Omega}{dr} \right)^2 \right]. \quad (43)$$

The fourth order solution, $W^{(4)}$, is obtained by making the substitution $W \rightarrow W^{(2)}$ in the right hand side of Eq.(38) and solving for W . Just as in point-splitting, this iteration procedure could continue indefinitely. However, in the present case of obtaining the $\vartheta(m^{-1})$ and $\vartheta(m^{-2})$ terms in the DeWitt-Schwinger expansion for $G^{(1)}$, the iterations need only be continued until the fourth order WKB solution, $W^{(4)}$, is reached. As pointed out by AHS [21], a second order WKB expansion contains the information of the $\vartheta(m^2)$ and $\vartheta(m^0)$ terms in the DeWitt-Schwinger point-splitting expansion, while a fourth order WKB expansion contains the information of the $\vartheta(m^{-2})$ terms. Thus, the infinite counterterms in the theory may be reproduced by performing the present calculation using only $W^{(2)}$, while the DeWitt-Schwinger approximations of the $\vartheta(m^{-1})$ and $\vartheta(m^{-2})$ terms require the use of $W^{(4)}$.

For purposes of simplicity and to make connections to previous work [21], the choice is made to split the points along the $\tau - \tau'$ direction such that $(x, x') \rightarrow (\tau, r, \theta, \phi; \tau', r, \theta, \phi)$. Making the definition [21]

$$A_1 = \lim_{r' \rightarrow r} \sum_{l=0}^{\infty} \left[2(l + \frac{1}{2}) p_{\omega l}(r) q_{\omega l}(r') \right] = \sum_{l=0}^{\infty} \left[\frac{(l + \frac{1}{2})}{r^2 W} \right], \quad (44)$$

with the limit $r' \rightarrow r$ being taken in the last expression, allows the unrenormalized Euclidean Green function to be written as

$$G_{E,unren}(x, \tau; x, \tau') = \langle \phi^2 \rangle_{unren} = \frac{1}{4\pi^2} \int_0^{\infty} d\omega \cos[\omega(\tau - \tau')] A_1, \quad (45)$$

for the separation along $\tau - \tau'$. The next section discusses the identification and removal of the two types of divergences that appear in this unrenormalized expression for $G_E(x, \tau; x, \tau')$.

III. RENORMALIZATION OF THE 4^{TH} ORDER WKB APPROXIMATION IN THE LARGE MASS LIMIT

There are two types of divergences which appear in Eq.(45). The first of these is a superficial divergence which appears when evaluating the sum over l with an upper limit of $l = \infty$, while the second is an ultraviolet divergence due to performing the integrations over

ω with an upper limit of $\omega = \infty$. This section presents the details of how the subtractions of these divergent terms from the unrenormalized Green function may be performed and how the resulting expression may be evaluated analytically.

As discussed more fully elsewhere [21,17,15,18], the superficial divergence due to the sums over l can not be real since $G_{E,unren}$ must remain finite with the points separated as they are in Eq.(45). The sum and integral present in Eq.(45) will be evaluated later by assuming the mass m of the quantum field is large enough to expand the summand of A_1 in inverse powers of m . At present, it is useful to consider how the false divergence over l is isolated assuming the sum and integral of Eq.(45) were to be evaluated without such a large mass expansion.

The superficial divergence that appears in A_1 may be identified by evaluating the sum

$$\sum_{l=0}^{\infty} 2(l + \frac{1}{2}) p_{\omega l} q_{\omega l} = \sum_{l=0}^{\infty} \frac{(l + \frac{1}{2})}{r^2 W} \quad (46)$$

in the limit $l \rightarrow \infty$. The only part of W that contributes as $l \rightarrow \infty$ is the zeroth order term of W , or Ω ; terms in W proportional to negative powers of Ω do not diverge as $l \rightarrow \infty$. Substituting $W = \Omega$ into $(l + \frac{1}{2})/r^2 W$ and expanding in inverse powers of l , and then truncating the expansion at $\vartheta(l^0)$ gives

$$\sum_{l=0}^{\infty} \lim_{l \rightarrow \infty} \frac{2(l + \frac{1}{2})}{2r^2 W} = \sum_{l=0}^{\infty} \frac{1}{r f^{\frac{1}{2}}}. \quad (47)$$

Thus, the counterterm $1/(r f^{\frac{1}{2}})$ must be subtracted from the summand of Eq.(44) in order to remove the superficial divergence over l . This now indicates the unrenormalized expression for $G_E(x, \tau; x, \tau')$ is given by

$$G_{E,unren}(x, \tau; x, \tau') = \frac{1}{4\pi^2} \int_0^{\infty} d\omega \cos[\omega(\tau - \tau')] \sum_{l=0}^{\infty} \left[\frac{(l + \frac{1}{2})}{r^2 W} - \frac{1}{r f^{\frac{1}{2}}} \right]. \quad (48)$$

This technique will serve to remove any superficial divergences over l that may appear. In Eq.(48), only the term $1/(r f^{\frac{1}{2}})$ is subtracted from the single summand $(l + \frac{1}{2})/r^2 W$. Later, when the WKB approximation for W is substituted into Eq.(45) and the summands and integrands are expanded in the large mass limit, many terms involving factors of l will arise.

Only some of those terms will contain a divergence in l , and those that do must have the appropriate counterterms in l subtracted from them to remove this superficial divergence.

Eq.(45) also contains the ultraviolet divergences that are known to arise in semiclassical quantum field theory. These divergences occur as $\omega \rightarrow \infty$ and must be subtracted from the unrenormalized expression. As in the case of the divergence over l , these subtractions must be performed term by term after the large mass expansions of A_1 have been performed. Thus, the generalized form for the renormalized expression for G_E is given by

$$G_{E,ren} = \frac{1}{4\pi^2} \int_0^\infty d\omega \cos[\omega(\tau - \tau')] \times \left[\sum_{l=0}^\infty \left\{ \frac{(l + \frac{1}{2})}{r^2 W} \right\}_{WKB \text{ expansion terms}} - (l \rightarrow \infty \text{ terms}) - (\omega \rightarrow \infty \text{ terms}) \right], \quad (49)$$

where the subscript “WKB expansion terms” indicates all of the terms that result from a 4th order WKB expansion of $(l + \frac{1}{2})/r^2 W$, and the terms identified as divergent in l and ω are subtracted from each of these terms of the expansion.

The calculation of all the terms in Eq.(49) is a straightforward but tedious process. The general procedure follows Appendices D, F, and G in the recent work of AHS [21] and will be described here. The presence of the gauge field increases the difficulty of the calculations substantially, but the method of AHS can be modified to account for the gauge field.

The calculation of $G_{E,ren}$ starts with the substitution of the 4th order WKB approximation for W into the definition of A_1 and the subsequent identification of the various $l \rightarrow \infty$ counterterms. To keep track of the order in the WKB expansions of quantities it is useful to introduce the dimensionless parameter α , letting $\alpha \rightarrow 1$ in the end. In Eq.(38), Ω is $\vartheta(\alpha^0)$, $V_1(r)$ is $\vartheta(\alpha^1)$, and the rest of the terms are $\vartheta(\alpha^2)$. The quantity $(l + \frac{1}{2})/(r^2 W)$ is expanded in powers of the WKB parameter α , truncating the expansions at $\vartheta(\alpha^4)$, and setting $\alpha = 1$. This will result in over 600 terms whose general form is given by

$$L_{hjk} = \sum_{l=0}^\infty \left[\frac{\omega^h 2(l + \frac{1}{2})^{1+2j}}{\Omega^k} - (l \rightarrow \infty \text{ terms}) \right], \quad (50)$$

where $\Omega(r)$ is defined in Eq.(39), and various factors of $f(r), V_1(r), eA_\tau(r), \dots$, that are present in front of each of the L_{hjk} have been omitted. This definition of L_{hjk} is similar to

that defined as L_{jk} in Ref. [21]. In the present work, the presence of the gauge field has led to the need to keep track of additional factors of ω which now enter into the definition of the L_{hjk} . The notation here is chosen so that it most closely matches the notation of AHS. Determining the $l \rightarrow \infty$ subtraction terms proceeds as before by expanding the summands in each L_{hjk} in inverse powers of l and truncating the expansion at $\vartheta(l^0)$. For example, the expression for L_{001} requires one subtraction term;

$$L_{001} = \sum_{l=0}^{\infty} \left[\frac{2(l + \frac{1}{2})}{\Omega} - \frac{2r}{f^{\frac{1}{2}}} \right]. \quad (51)$$

Determining the $\omega \rightarrow \infty$ subtraction terms requires using the Plana sum formula [21,25]

$$\sum_{l=k}^{\infty} g(l) = \frac{1}{2}g(k) + \int_k^{\infty} g(\tau)d\tau + i \int_0^{\infty} \frac{dt}{e^{2\pi t} - 1} [g(k + it) - g(k - it)]. \quad (52)$$

The first two terms of Eq.(52) may be computed analytically, while the third term is not in general able to be computed analytically. Since the ultraviolet, or $\omega \rightarrow \infty$, behavior of these sums is dominant, the third term of Eq.(52) may be expanded in inverse powers of ω . The expansions are truncated at $\vartheta(\omega^{-1})$ since terms of $\vartheta(\omega^{-3})$ are not ultraviolet divergent [21]. Then, the integrals that arise from the expansion in inverse powers of ω may be computed analytically. Applying this procedure to L_{001} , for example, gives

$$\begin{aligned} L_{001} &= \sum_{l=0}^{\infty} \left[\frac{2(l + \frac{1}{2})}{\Omega} - \frac{2r}{f^{\frac{1}{2}}} \right] \\ &= \frac{1}{2 \left[(\omega - eA_{\tau})^2 + m^2 f + \frac{f}{4r^2} \right]^{1/2}} \\ &\quad - \frac{2r^2}{f} \left[(\omega - eA_{\tau})^2 + m^2 f + \frac{f}{4r^2} \right]^{1/2} \\ &\quad - \frac{4}{\omega} \int_0^{\infty} dt \frac{t}{e^{2\pi t} - 1} + \dots \\ &= -\frac{2r^2}{f} \omega + \frac{2er^2 A_{\tau}}{f} + \left(\frac{1}{12} - m^2 r^2 \right) \frac{1}{\omega} + \vartheta(\omega^{-3}). \end{aligned} \quad (53)$$

The only L_{hjk} for which there will be ω subtraction terms are those for which $\{j, k\}$ have the values $\{0, 1\}$, $\{0, 3\}$, $\{1, 3\}$, $\{1, 5\}$, $\{2, 5\}$ and $\{2, 7\}$; only $h = 0$ occurs for these combinations of $\{j, k\}$. Comparison of L_{001} in this example with L_{01} of AHS, along with Eq.(25), show how

the gauge field will bring complex quantities into the expressions for the L_{hjk} and eventually into $G^{(1)}$.

Once the terms divergent in both l and ω have been determined for each of the L_{hjk} , the expression for $G_{E,ren}$ is now given by

$$G_{E,ren} = \frac{1}{4\pi^2} \int_0^\infty d\omega \cos[\omega(\tau - \tau')] \{F(r)(L_{hjk} - L_{hjk,\omega \rightarrow \infty})\}, \quad (54)$$

where each of the $F(r)$ above represent the functions such as $f(r), V_1(r), eA_\tau(r), \dots$, that appear in the calculation but are not affected by the summations and integrations, and the curly brackets $\{\dots\}$ indicate the large number of terms arising due to the WKB expansions. In Eq.(54), the $\omega \rightarrow \infty$ counterterms for each L_{hjk} are indicated symbolically by $L_{hjk,\omega \rightarrow \infty}$. Since the terms which diverge as $\omega \rightarrow \infty$ are subtracted in Eq.(54), then the function $\cos[\omega(\tau - \tau')]$ will be dominated by the infinitesimal separation $(\tau - \tau')$. This indicates that $\cos[\omega(\tau - \tau')] \approx 1$ is valid. For the purposes of completeness, it should be noted that future work involving both $\langle j^\mu \rangle$ and $\langle T_{\mu\nu} \rangle$ will involve more factors of ω than those that arise within the L_{hjk} . Thus, it is useful to make the definition

$$S_{ihjk} = \frac{1}{4\pi^2} \int_0^\infty d\omega \omega^i (L_{hjk} - L_{hjk,\omega \rightarrow \infty}), \quad (55)$$

where the indices $ihjk$ have been chosen so the present work is more easily compared to that of AHS. In the calculation of the Green function in this paper, $i = 0$ throughout; in AHS, $i = 0$ for the Green function calculation while $i = 0$ or 2 when calculating $\langle T_{\mu\nu} \rangle$. The sums and integrals in Eq.(55) still can not be evaluated analytically in general in their present form. They may be put into a form that can be evaluated analytically by taking the large mass limit of the quantum field.

Each of the S_{ihjk} may be computed in the large mass limit by first expanding each $\omega^{i+2(l+\frac{1}{2})}/\Omega^k$ term in the L_{hjk} using the Plana sum formula of Eq.(52). The first two terms of the Plana formula may be computed exactly. Again, the third term may not be computed analytically, but once expanded in the large mass limit each piece of the expansion may be computed analytically. The large mass limit of this third term is

obtained by expanding the integrand of the third term in inverse powers of the large quantity $\sqrt{(\omega - eA_\tau)^2 + m^2 f + (l + \frac{1}{2})^2 f/r^2}$. Each integral may then be evaluated analytically. After the integrations, the entire expression for each S_{ihjk} is expanded in inverse powers of m . In Appendix G of Ref. [21], AHS give an example of this type of calculation when they calculate the large mass expansion of S_{001} . Here, the large mass expansion of S_{0001} will be shown for comparison.

The $l \rightarrow \infty$ limit for L_{001} is found in Eq.(51), while the $\omega \rightarrow \infty$ subtraction term is found in Eq.(53). This gives the expression to be evaluated in the large mass limit as

$$S_{0001} = \frac{1}{4\pi^2} \int_0^\infty d\omega \omega^0 \times \left\{ \sum_{l=0}^\infty \left[\left(\frac{2(l + \frac{1}{2})}{[(\omega - eA_\tau)^2 + m^2 f + (l + \frac{1}{2})^2 \frac{f}{r^2}]^{1/2}} - \frac{2r}{f^{1/2}} \right) - \left(-\frac{2r^2}{f} \omega + \left(\frac{1}{12} - m^2 r^2 \right) \frac{1}{\omega} + \frac{2er^2 A_\tau}{f} \right) \right] \right\}. \quad (56)$$

Using the Plana sum formula and expanding in inverse powers of the large quantity $\sqrt{(\omega - eA_\tau)^2 + m^2 f + (l + \frac{1}{2})^2 f/r^2}$ yields

$$S_{0001} = \frac{1}{4\pi^2} \int_0^\infty d\omega \left\{ \left(\frac{1}{2} \right) \left(\frac{1}{[(\omega - eA_\tau)^2 + m^2 f + (l + \frac{1}{2})^2 \frac{f}{r^2}]^{1/2}} - \frac{2r}{f^{1/2}} \right) + \int_0^\infty dl \left(\frac{1}{[(\omega - eA_\tau)^2 + m^2 f + (l + \frac{1}{2})^2 \frac{f}{r^2}]^{1/2}} - \frac{2r}{f^{1/2}} \right) + i \int_0^\infty \frac{dt}{e^{2\pi t} - 1} \left\{ \frac{2(it + \frac{1}{2})}{[(\omega - eA_\tau)^2 + m^2 f + (it + \frac{1}{2})^2 \frac{f}{r^2}]^{1/2}} - \frac{2(-it + \frac{1}{2})}{[(\omega - eA_\tau)^2 + m^2 f + (-it + \frac{1}{2})^2 \frac{f}{r^2}]^{1/2}} \right\} - \left(-\frac{2r^2}{f} \omega + \left(\frac{1}{12} - m^2 r^2 \right) \frac{1}{\omega} + \frac{2er^2 A_\tau}{f} \right) \right\}, \quad (57)$$

$$S_{0001} = \frac{1}{4\pi^2} \int_0^\infty d\omega \left\{ -\frac{2r^2}{f} [(\omega - eA_\tau)^2 + m^2 f + \frac{f}{4r^2}]^{1/2} + \frac{1}{[(\omega - eA_\tau)^2 + m^2 f + \frac{f}{4r^2}]^{1/2}} + \frac{f}{30r^2 [(\omega - eA_\tau)^2 + m^2 f + \frac{f}{4r^2}]^{3/2}} \right\}$$

$$\begin{aligned}
& + \frac{f^2}{105r^4[(\omega - eA_\tau)^2 + m^2f + \frac{f}{4r^2}]^{5/2}} + \dots \\
& - \left(-\frac{2r^2}{f}\omega + \left(\frac{1}{12} - m^2r^2 \right) \frac{1}{\omega} + \frac{2er^2A_\tau}{f} \right) \Bigg\}.
\end{aligned} \tag{58}$$

Performing the integrals over ω and expanding in the large mass limit yields

$$\begin{aligned}
S_{0001} \approx & \frac{m^2r^2}{8\pi^2} \left[-1 + \ln \left(\frac{m^2f}{4\lambda^2} \right) \right] - \frac{mr^2eA_\tau}{2\pi^2f^{\frac{1}{2}}} \\
& - \frac{1}{96\pi^2} \ln \left(\frac{m^2f}{4\lambda^2} \right) - \frac{r^2(eA_\tau)^2}{4\pi^2f} \\
& + \frac{1}{m} \left(\frac{eA_\tau}{48\pi^2f^{\frac{1}{2}}} - \frac{r^2(eA_\tau)^3}{12\pi^2f^{\frac{3}{2}}} \right) + \frac{7}{3840m^2\pi^2r^2} \\
& + \frac{1}{m^3} \left(\frac{7eA_\tau}{3840\pi^2r^2f^{\frac{1}{2}}} - \frac{(eA_\tau)^3}{288\pi^2f^{\frac{3}{2}}} + \frac{r^2(eA_\tau)^5}{80\pi^2f^{\frac{5}{2}}} \right) \\
& + \frac{31}{64512m^4\pi^2r^4} + \vartheta(m^{-5}).
\end{aligned} \tag{59}$$

In Eq.(59), the infrared cutoff parameter λ has been introduced due to the lower limit $\omega = 0$ on the integrals over ω [2,21]. Comparison of Eq.(59) with Eq.(G2) of Ref. [21] shows that the presence of the gauge field has brought odd powers of m into the asymptotic expansions for the S_{ihjk} . Fortunately, when the large mass expansions of all of the S_{ihjk} are computed, many are found to be of such a high order in inverse powers of m that they do not contribute to the final expression for $G_{E,ren}$ since the DeWitt-Schwinger expansions are truncated at $\vartheta(m^{-2})$. Note that the gauge field has also contributed terms of $\vartheta(m^1)$ and $\vartheta(m^0)$ to S_{0001} , as it does for some of the other S_{ihjk} . In this work, only terms which contain negative powers of m will be considered in the final DeWitt-Schwinger approximation. Finally, all of the S_{ihjk} are substituted into the general expression for $G_{E,ren}$

$$G_{E,ren} = \{F(r)S_{ihjk}\}, \tag{60}$$

where the curly brackets $\{\dots\}$ indicate the great number of terms to be added together, and like powers of m are collected.

IV. RESULTS

When all the substitutions are made and terms of like powers of m are collected, the result is that the coefficients of the negative powers of m constitute the DeWitt-Schwinger approximation to $G_{E,ren}$ for a complex scalar field in a static spherically symmetric space-time. A complex scalar field was chosen in order to make a connection to the earlier work of DeWitt [9,6] and Christensen [10] involving a real scalar field since the results of the present work must reduce to their results when the charge of the field vanishes. In their work, the generalized DeWitt-Schwinger approximation for the Hadamard elementary function for a scalar field is

$$\frac{1}{2}G^{(1)} = \langle \phi^2 \rangle \approx \frac{[a_2]}{16\pi^2 m^2}, \quad (61)$$

where $[a_2]$ is given by [9,24]

$$\begin{aligned} [a_2] = & -\frac{1}{180}R^{\alpha\beta}R_{\alpha\beta} + \frac{1}{180}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} + \frac{1}{6}\left(\frac{1}{5} - \xi\right)R_{;\alpha}{}^{\alpha} \\ & + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2 - \frac{e^2}{12}F^{\alpha\beta}F_{\alpha\beta}. \end{aligned} \quad (62)$$

In the present work, the m^{-2} term in the DeWitt-Schwinger approximation for $G_{E,ren} = \frac{1}{2}G^{(1)} = \langle \phi^2 \rangle$ is given in Euclidean space by

$$\begin{aligned} G_{E,m^{-2}} = & \frac{1}{m^2 \pi^2} \left(\frac{1}{240 r^4} - \frac{1}{240 r^4 h^2} + \frac{\xi^2 R^2}{32} \right. \\ & - \frac{e^2 A_\tau'^2}{96 f h} + \frac{f'}{96 r^3 f h^2} - \frac{f'}{96 r^3 f h} + \frac{23 f'^2}{5760 r^2 f^2 h^2} \\ & + \frac{f'^2}{576 r^2 f^2 h} - \frac{23 f'^3}{1920 r f^3 h^2} + \frac{7 f'^4}{1280 f^4 h^2} - \frac{3 h'}{160 r^3 h^3} \\ & + \frac{h'}{96 r^3 h^2} - \frac{f' h'}{96 r^2 f h^3} + \frac{f' h'}{576 r^2 f h^2} - \frac{f'^2 h'}{80 r f^2 h^3} \\ & + \frac{13 f'^3 h'}{1920 f^3 h^3} - \frac{3 h'^2}{640 r^2 h^4} - \frac{11 f' h'^2}{640 r f h^4} + \frac{5 f'^2 h'^2}{768 f^2 h^4} \\ & + \frac{7 h'^3}{240 r h^5} + \frac{7 f' h'^3}{960 f h^5} + \frac{f''}{288 r^2 f h^2} - \frac{f''}{288 r^2 f h} \\ & \left. + \frac{7 f' f''}{320 r f^2 h^2} - \frac{13 f'^2 f''}{960 f^3 h^2} + \frac{13 h' f''}{960 r f h^3} - \frac{5 f' h' f''}{384 f^2 h^3} \right) \end{aligned}$$

$$\begin{aligned}
& -\frac{19 h'^2 f''}{1920 f h^4} + \frac{f''^2}{192 f^2 h^2} + \frac{h''}{240 r^2 h^3} + \frac{f' h''}{120 r f h^3} \\
& -\frac{f'^2 h''}{384 f^2 h^3} - \frac{13 h' h''}{480 r h^4} - \frac{13 f' h' h''}{1920 f h^4} + \frac{f'' h''}{240 f h^3} \\
& + \xi \left\{ -\frac{R^2}{96} + \left(-\frac{1}{48 r h} - \frac{f'}{192 f h} + \frac{h'}{192 h^2} \right) R' - \frac{R''}{96 h} \right\} \\
& -\frac{f'''}{120 r f h^2} + \frac{f' f'''}{192 f^2 h^2} + \frac{h' f'''}{160 f h^3} + \frac{h'''}{240 r h^3} \\
& + \frac{f' h'''}{960 f h^3} - \frac{f'''}{480 f h^2} \Big) \tag{63}
\end{aligned}$$

where the primes indicate differentiation with respect to r . This is exactly the same result that is obtained by writing Eqs.(61)-(62) in terms of the metric functions of Eq.(22). Since the gauge field only appears here in the form $(eA'_r)^2$, rotating back to Lorentzian space to obtain $G^{(1)}$ simply results in the appearance of a negative sign for this term. This demonstrates the correspondence between the present method in a static spherically symmetric spacetime and the method of the generalized DeWitt-Schwinger expansion; they yield the same information for the real part of $G^{(1)}$ in the limit that the mass of the quantum field is large when compared to the inverse of the radius of curvature of the spacetime.

With the information in Eq.(63) previously known and accepted, it is fortunate that the present method not only duplicates that information but also yields more information. The DeWitt-Schwinger point-splitting expansion is fully covariant and yields complete information about each of the coefficients in the expansion, indicating that all of the terms in a large mass DeWitt-Schwinger expansion proportional to inverse, even powers of m have been determined by the generalized expansion. These terms are constructed out of real curvature and electromagnetic field tensors and thus are incapable of yielding the imaginary part of $G^{(1)}$. Yet the definition of $G^{(1)}$ for a complex scalar field,

$$G^{(1)}(x, x') \equiv \langle 0 | \{ \phi(x), \phi^*(x') \} | 0 \rangle, \tag{64}$$

shows that it is, in general, a complex quantity. Any method for calculating $G^{(1)}$ that does not yield both real and imaginary pieces is incomplete in its determination of $G^{(1)}$. DeWitt emphasized this limitation existed for the point-splitting procedure by explicitly stating that

the expansion in inverse powers of m^2 was incapable of yielding the imaginary part of $G^{(1)}$ [6]. Yet this imaginary part is non-vanishing even in flat space with a constant gauge field. Classical field theory calculations of the current due to a complex scalar field under these conditions is not identically zero [26]. Schwinger's work [7] incorporated these flat space, constant field conditions specifically with the result being the non-vanishing Schwinger pair creation rate of Eq.(5).

The only way in which the imaginary part of $G^{(1)}$ may be obtained in a DeWitt-Schwinger expansion is by obtaining the terms proportional to inverse, odd powers of m . It is possible to anticipate the form of the terms in the expansion for $G^{(1)}$. Moving to units where $c = \hbar = 1$ but $G \neq 1$, then $G^{(1)}$ and curvature tensors such as $R_{\mu\nu}$ are second order quantities with units of $(length)^{-2}$ while m and A_τ have units of $(length)^{-1}$. Using such power counting, Davies *et al.* [27] constructed the stress energy tensor of a conformally invariant scalar field in a conformally invariant spacetime. In the present case of the expansion for $G^{(1)}$ the same procedure may be used. The term proportional to m^{-1} must be proportional to $(length)^{-3}$ in order for $G^{(1)}$ to remain second order. Since Eq.(64) holds in both flat and curved spacetimes, the present method should yield a non-vanishing imaginary part when $f(r) = h(r) = 1$ and when the gauge field $A_\tau = \text{constant}$. Under these conditions, the m^{-1} term of G_E should be proportional to $(eA_\tau)^3$ and $(eA_\tau)r^{-2}$. This power counting may be expanded by allowing the gauge field to be a function of r , with the result that additional terms such as $(eA'_\tau)r^{-1}$ should appear in the expansion.

The m^{-1} term in the DeWitt-Schwinger approximation for $G_{E,ren}$ in the present work is given in Euclidean space by

$$\begin{aligned}
G_{E,m^{-1}} = \frac{1}{m \pi^2} & \left(-\frac{(e A_\tau)^3}{24 f^{\frac{3}{2}}} + \frac{e A_\tau}{24 r^2 f^{\frac{1}{2}}} - \frac{e R \xi A_\tau}{8 f^{\frac{1}{2}}} \right. \\
& - \frac{e A_\tau}{24 r^2 f^{\frac{1}{2}} h} + \frac{e A'_\tau}{24 r f^{\frac{1}{2}} h} - \frac{e A_\tau f'}{16 r f^{\frac{3}{2}} h} - \frac{e A'_\tau f'}{48 f^{\frac{3}{2}} h} \\
& + \frac{3 e A_\tau f'^2}{128 f^{\frac{5}{2}} h} + \frac{e A_\tau h'}{24 r f^{\frac{1}{2}} h^2} - \frac{e A'_\tau h'}{96 f^{\frac{1}{2}} h^2} + \frac{e A_\tau f' h'}{64 f^{\frac{3}{2}} h^2} \\
& \left. + \frac{e A''_\tau}{48 f^{\frac{1}{2}} h} - \frac{e A_\tau f''}{32 f^{\frac{3}{2}} h} \right). \tag{65}
\end{aligned}$$

All of these terms are proportional to odd powers of A_τ and its derivatives and, due to Eq.(25), are thus all imaginary when rotated back to Lorentzian space. All of these terms are also proportional to odd powers of the charge e of the quantum field. Upon imposing charge conjugation symmetry on $\langle \phi^2 \rangle = \frac{1}{2}G^{(1)}$ [28], these terms will not contribute to the calculation of the vacuum polarization in the spacetime.

It is remarkable that the two methods yield exactly the same result for the m^{-2} term in their respective DeWitt-Schwinger expansions. As pointed out by Gibbons [29], while the DeWitt-Schwinger point-splitting expansion is widely used in renormalization theory, there is still much that is unknown about the type of series expansion that it really is. The nonzero results of Eq.(65) highlight the major difference between the present method and the generalized DeWitt-Schwinger point-splitting expansion. A major limitation of the DeWitt-Schwinger point-splitting expansion is that it is an asymptotic expansion limited to inverse powers of m^2 . This dictates that it is restricted to the calculation of only the real part of $G^{(1)}$ which, as stated above, is generally complex. Yet the DeWitt-Schwinger expansion has the distinct advantage of being a completely general expansion with its pieces constructed from curvature and electromagnetic tensors. Thus, once the point-splitting calculations have been performed and renormalization achieved by discarding terms proportional to nonnegative powers of m , the remainder is a completely general expression which may then be used to study any spacetime for which a metric exists.

The main limitation of the present method is that it is restricted to spacetimes for which the basis functions of the quantum field can be put into the WKB form like that of Eq.(38). The wave equation must therefore be separable. Since this has been achieved for the Kerr metric [30], the present work could conceivably be extended to Kerr spacetimes.

The major advantage of the present method over the DeWitt-Schwinger point-splitting expansion is that it can yield the imaginary part of $G^{(1)}$. The calculation of the imaginary part of $G^{(1)}$ is important due to the definition of the current $\langle j^\mu \rangle$ in Eq.(13), which is non-vanishing even in flat space with a constant gauge field. The factor of i in front of the expression, along with the subtraction of the complex conjugate of the derivatives of $G^{(1)}$,

show that only the imaginary part of $G^{(1)}$ will contribute to the current. The factor of e in front of the expression for $\langle j^\mu \rangle$ requires the presence of the odd powers of e in the imaginary part of $G^{(1)}$ if the current is to be constructed from $G^{(1)}$. The final expression for the current must be proportional to even powers of e if the current is not to vanish under charge conjugation symmetry [28]. Work is in progress to calculate the DeWitt-Schwinger approximation for the current in two ways; first, directly from Eqs.(65) and (13), and second, starting from Eq.(13) and proceeding through the steps outlined in this paper to calculate the DeWitt-Schwinger approximations for quantities such as $G^{(1)|\mu}$ that are required to construct $\langle j^\mu \rangle$.

Much of the previous research in quantum field theory in curved spacetimes dealt with the production of virtual particles due to vacuum polarization. Whenever real particle production due to an electromagnetic field was to be studied, investigators most often used the pair creation rate derived by Schwinger for flat spacetime [31]. While Schwinger's work assumed a constant electric field E , the authors in Ref. [31] replaced that field with the spherically symmetric radial field of the Reissner-Nordström spacetime. With the curved space results in Eq.(65), it should now be possible to determine how the effects of curved space will cause the rate of charged particle production to differ from that of the flat space Schwinger rate.

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